On embeddings of grand grand Sobolev-Morrey spaces with dominant mixed derivatives

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Abstract

In this paper it is constructed a new grand grand Sobolev-Morrey $S^l_{p),\varkappa),a,\alpha}W(G)$ spaces with dominant mixed derivatives. With help integral representation of generalized mixed derivatives of functions, defined on n-dimensional domains satisfying flexible horn condition, an embedding theorem is proved. In other works, the embedding theorem is proved in these spaces and belonging of the generalized mixed derivatives of functions from these spaces to the Holder class, was studied.

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1 Introduction and preliminary notes

The fact some mixed derivatives $D^\nu f$ cannot be estimated in terms of the derivatives of f entering the definition of the norm of W^l_p, H^l_p and $B^l_{p,\theta}$, leads to the necessity of consideration of the function spaces of another type in which the key role is played by mixed derivatives.

The functions spaces Sobolev $S^l_pW(G)(l \in N^n)$ and Nikolskii $S^l_pH(G)$ with dominant mixed derivatives were introduced and studied by S.M. Nikolskii [19] and later by T.I. Amanov [1] and A.D. Djabrailov [4] were introduced and studied the spaces Nikolskii-Besov $S^l_{p,\theta}B(G)$ with dominant mixed derivatives. The spaces type Sobolev-Morrey $S^l_{p,a,\varkappa,\tau}W(G)$ with dominant mixed derivatives were introduced and studied in [15].

Example 1.1. Let us consider an equation of the form

$$u^{(2)}_{xy} + u^{(1)}_x + u^{(1)}_y + u = f(x),$$

in our case the solution of this equation is sought in the space $S^{(1,1)}W$. One can look for the solution of equation in the space $W^{(2,2)}$, but then this solution will require additional derivatives, in other words, in our case the solution belongs to a wider class.

In this paper we introduce a grand grand Sobolev-Morrey $S^l_{p),\varkappa),a,\alpha}W(G)$ spaces with dominant mixed derivatives and studied differential and differential-difference properties of functions from this spaces.

It should be noted that the grand Lebesgue spaces $L_{p)}(G)$, $1 < p < \infty$, on bounded domain $G \subset R^n$ was introduced by T. Iwaniec and C. Sbordone [5] and the small Lebesgue spaces, the grand grand Lebesgue-Morrey spaces, the grand and grand grand Sobolev-Morrey spaces type (with

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different norms) were introduced and studied in [3], [7]-[14], [17], [18], [20]-[22]. Note that in this paper, not only does the class expand, but also it is proved that in the "Hölder index" is greater than in the papers [6, 15, 16].

Let $G \subset R^n$ be a bounded domain, $e_n = \{1, 2, \ldots, n\}$, $e \subseteq e_n$, $l = (l_1, \ldots, l_n)$, $l_j > 0$ $(j \in e_n)$ are integers, and let $l^e = (l_1^e, \ldots, l_n^e)$, $l_j^e = l_j$ for $j \in e$; $l_j^e = 0$ for $j \in e_n \setminus e = e'$. Let $t = (t_1, \ldots, t_n)$, $t_j > 0$ $(j \in e_n)$ and for $x \in R^n$ we put

$$G_{t^{\varkappa}}(x) = G \cap I_{t^{\varkappa}}(x) = G \cap \left\{ y : |y_j - x_j| < \frac{1}{2} t_j^{\varkappa_j},\ j \in e_n \right\},$$

and

$$\int_{a^e}^{b^e} f(x)dx^e = \left(\prod_{j\in e} \int_{a_j}^{b_j} dx_j \right) f(x),$$

i.e. integration is carried out only with respect to the variables x_j whose indices belong to e.

Definition 1.2. Denote by $S^l_{p),\varkappa),a,\alpha}W(G)$ a space of locally summable on G having the generalized derivatives $D^{l^e} f (e \subseteq e_n)$ with the finite norm

$$\|f\|_{S^l_{p),\varkappa),a,\alpha}W(G)} = \sum_{e\subseteq e_n} \|D^{l^e} f\|_{p),\varkappa),a,\alpha;G}, \tag{1.1}$$

where

$$\|f\|_{p),\varkappa),a,\alpha;G} = \|f\|_{L_{p),\varkappa),a,\alpha}(G)}$$
$$= \sup_{\substack{x\in G,\\ 0<t_j\le d_j,\\ 0<\varepsilon<s_m}} \left(\frac{1}{\prod_{j\in e_n} t_j^{\varkappa_j a - \alpha_j \varepsilon}} \frac{\varepsilon}{|G^{(x)}_{t^{\varkappa}}|} \int_{G_{t^{\varkappa}}(x)} |f(y)|^{p-\varepsilon} dy \right)^{\frac{1}{p-\varepsilon}}, \tag{1.2}$$

d_j $(j \in e_n)$ is diagonals of $I_{t^{\varkappa}}(x)$, $s_m = \min\{s_1, ..., s_n\}$, $s_j = \min\left\{p - 1, \frac{\varkappa_j a}{\alpha_j}\right\}$, $\alpha_j \ge 0$ $(j \in e_n)$; $1 < p < \infty$, $\varkappa \in (0, \infty)^n$; $a \in [0, 1]$(suppose that $\frac{0}{0}$ equal 0).

We note some properties of the spaces $L_{p),\varkappa),a,\alpha}(G)$ and $S^l_{p),\varkappa),a,\alpha}W(G)$.

1. For any $\varkappa, \alpha \in (0, \infty)^n$, $a \in [0, 1]$ we have the embedding

$$L_{p),\varkappa),a,\alpha}(G) \hookrightarrow L_{p)}(G),\ S^l_{p),\varkappa),a,\alpha}W(G) \hookrightarrow S^l_{p)}W(G)$$

i.e.

$$\|f\|_{p),G} \le \|f\|_{p),\varkappa),a,\alpha,G} \quad \text{and} \quad \|f\|_{S^l_{p)}W(G)} \le \|f\|^l_{S_{p),\varkappa),a,\alpha}W(G)}, \tag{1.3}$$

$$\|f\|_{S^l_pW(G)} = \sum_{e\subseteq e_n} \|D^{l^e} f\|_{p),G}, \quad \|f\|_{p),G} = \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon}{|G|} \int_G |f(x)|^{p-\varepsilon} dx \right)^{\frac{1}{p-\varepsilon}}.$$

2. The spaces $L_{p),\varkappa),a,\alpha}(G)$ and are complete $S^l_{p),\varkappa),a,\alpha}W(G)$ are complete. The proof scheme completeness of these spaces is carried out as in [2, p.398]
3.

$$\|f\|_{p),\varkappa),0,0;G} = \|f\|_{p);G} \text{ and } \|f\|_{S^l_{p),\varkappa),0,0}W(G)} = \|f\|_{S^l_{p)}W(G)}.$$

Let $M(\cdot, y, z) \in C_0^\infty$ so as to have

$$S(M) = \bigcup_{\substack{0<t_j\le T_j,\\ j\in e_n}} \left\{ y : \left(\frac{y}{t^e + T^{e'}}\right) \in S(M) \right\},$$

where $t^e + T^{e'} = t_j (j \in e)$, $t^e + T^{e'} = T_j \ (j \in e')$ and clearly, $V \subset I_T = \left\{ x : |x_j| < \frac{1}{2}T_j,\ j \in e_n \right\}$. Let V be an open set contained in the domain G, henceforth we always assume that $U + V \subset G$, and put

$$G_{T^\varkappa}(V) = (U + I_{T^\varkappa}(x)) \cap G.$$

Obviously, if $0 < \varkappa_j \le 1$ $(j \in e_n)$, then $I_T \subset I_{T^\varkappa}$ and thereby $U + V \subset G_{T^\varkappa}(V) = Z$.

Lemma 1.3. Let $1 < p < q \le r \le \infty$, $\varkappa_j \le \frac{1}{1+a}$, $0 < t_j$, $\eta_j \le T_j \le \min\{1, d_j\}$, $0 < \gamma_j < \gamma_{j,0}$, $\nu = (\nu_1, \ldots, \nu_n)$, $\nu_j \ge 0$ are integers $(j \in e_n)$, $\psi \in L_{p),\varkappa),a,\alpha}(G)$ and

$$\mu_j = l_j - \nu_j - (1 - \varkappa_j - \varkappa_j a + \alpha_j \varepsilon) \left(\frac{1}{p - \varepsilon} + \frac{1}{q - \varepsilon} \right),$$

$$A^e_\eta(x) =$$

$$= \int\limits_{O^e}^{\eta^e} \prod_{j\in e} t_j^{l_j - \nu_j - 2} \int\limits_{R^n} \psi(x + y) M \left(\frac{y}{t^e + T^{e'}}, \frac{\rho(t^e + T^{e'}, x)}{t^e + T^{e'}}, \rho'(t^e + T^{e'}, x) \right) dy dt^e \tag{1.4}$$

$$A^e_{\eta,T}(x) =$$

$$= \int\limits_{\eta^e}^{T^e} \prod_{j\in e} t_j^{l_j - \nu_j - 2} \int\limits_{R^n} \psi(x + y) M \left(\frac{y}{t^e + T^{e'}}, \frac{\rho(t^e + T^{e'}, x)}{t^e + T^{e'}}, \rho'(t^e + T^{e'}, x) \right) dy dt^e. \tag{1.5}$$

Then

$$\sup_{\overline{x}\in U} \|A^e_\eta\|_{q-\varepsilon, U_{\gamma^\varkappa}(\overline{x})} \le C_1 \|\psi\|_{p),\varkappa),a,\alpha;G} \varepsilon^{-\frac{1}{p-\varepsilon}} \prod_{j\in e_n} \gamma_j^{\frac{\varkappa_j a + \varkappa_j - \alpha_j \varepsilon}{q - \varepsilon}}$$
$$\times \prod_{j\in e'} T_j^{1-(1-\varkappa_j - \varkappa_j a + \alpha_j \varepsilon)\left(\frac{1}{p-\varepsilon} - \frac{1}{q-\varepsilon}\right)} \prod_{j\in e} \eta_j^{\mu_j} \quad (\mu_j > 0), \tag{1.6}$$

$$\sup_{\overline{x}\in U}\|A^e_{\eta,T}\|_{q-\varepsilon,U_{\gamma^x}(\overline{x})}\le C_2\|\psi\|_{p),\varkappa),a,\alpha;G}\varepsilon^{-\frac{1}{p-\varepsilon}}\prod_{j\in e_n}\gamma_j^{\frac{\varkappa_j a+\varkappa_j-\alpha_j\varepsilon}{q-\varepsilon}}$$

$$\times\prod_{j\in e'}T^{1-(1-\varkappa_j-\varkappa_j a+\alpha_j\varepsilon)\left(\frac{1}{p-\varepsilon}-\frac{1}{q-\varepsilon}\right)}\begin{cases}\prod\limits_{j\in e}T_j^{\mu_j}, & \text{for } \mu_j>0\\ \prod\limits_{j\in e}\ln\frac{T_j}{\eta_j}, & \text{for } \mu_j=0\\ \prod\limits_{j\in e}\eta_j^{\mu_j}, & \text{for } \mu_j<0\end{cases}. \tag{1.7}$$

Here $U_{\gamma^\varkappa}(\overline{x})=\{x:|x_j-\overline{x}_j|<\frac{1}{2}\gamma_j^{x_j},\ j\in e_n\}$, and C_1 and C_2 are constants independent ψ, γ, η, T and ε.

Proof. Using the generalized Minkowskii inequality for any $\overline{x}\in U$ and $0<\varepsilon<s_m$ we have

$$\|A^e_\eta\|_{q-\varepsilon,U_{\gamma^\varkappa}(\overline{x})}\le\int\limits_{O^e}^{\eta^e}\prod_{j\in e}t_j^{l_j-\nu_j-2}\|\varphi(\cdot,t^e+T^{e'})\|_{q-\varepsilon,U_{\gamma^\varkappa}(\overline{x})}dt^e, \tag{1.8}$$

where

$$\varphi(x,t^e+T^{e'})=\int\limits_{R^n}\psi(x+y)M\left(\frac{y}{t^e+T^{e'}},\frac{\rho(t^e+T^{e'},x)}{t^e+T^{e'}},\rho'\left(t^e+T^{e'},x\right)\right)dy\ . \tag{1.9}$$

Estimate of the norm $\|\varphi(\cdot,t^e+T^{e'})\|_{q-\varepsilon,U_{\gamma^\varkappa}(\overline{x})}$. From Hölder's inequality $(q\le r)$ we obtain

$$\|\varphi(\cdot,t^e+T^{e'})\|_{q-\varepsilon,U_{\gamma^\varkappa}(\overline{x})}\le\|\varphi(\cdot,t^e+T^{e'})\|_{r-\varepsilon,U_{\gamma^\varkappa}(\overline{x})}\prod_{j\in e_n}\gamma_j^{\varkappa_j\left(\frac{1}{q-\varepsilon}-\frac{1}{r-\varepsilon}\right)}. \tag{1.10}$$

Let X be the characteristic function of $S(M)$. Using the fact that $1\le p\le r\le\infty$, $s\le r\left(\frac{1}{s}=1-\frac{1}{p-\varepsilon}+\frac{1}{r-\varepsilon}\right)$ and

$$|\psi M|=(|\psi|^{p-\varepsilon}|M|^s)^{\frac{1}{r-\varepsilon}}(|\psi|^{p-\varepsilon}\varkappa)^{\frac{1}{p-\varepsilon}-\frac{1}{r-\varepsilon}}(|M|^s)^{\frac{1}{s}-\frac{1}{r-\varepsilon}}$$

and applying again Hölder's inequality $\left(\frac{1}{r-\varepsilon}+\left(\frac{1}{p-\varepsilon}-\frac{1}{r-\varepsilon}\right)+\left(\frac{1}{s}-\frac{1}{r-\varepsilon}\right)=1\right)$, we have

$$\|\varphi(\cdot,t^e+T^{e'})\|_{r-\varepsilon,U_{\gamma^*}(\overline{x})}\le K\sup_{(\overline{x})\in U_{\gamma^*}(\overline{x})}\left(\int\limits_{R^n}|\psi(x+y)|^{p-\varepsilon}\varkappa\left(\frac{y}{t^e+T^{e'}}\right)dy\right)^{\frac{1}{p-\varepsilon}-\frac{1}{r-\varepsilon}}$$

$$\times\sup_{y\in V}\left(\int\limits_{U_{\gamma^\varkappa}(\overline{x})}|\psi(x+y)|^{p-\varepsilon}dx\right)^{\frac{1}{r-\varepsilon}}\left(\int\limits_{R^n}\left|M_1\left(\frac{y}{t^e+T^{e'}}\right)\right|^s dy\right)^{\frac{1}{s}}, \tag{1.11}$$

here suppose that $|M(x,y,z)|\le K|M_1(x)|$.

Obviously, if $\varkappa_j \le 1$, $0 < t_j \le T_j (j \in e_n)$, then $Z_{t^e+T^{e'}}(x) \subset Z_{(t^{\varkappa})^e+(T^{\varkappa})^{e'}}(x)$. For every $x \in U$ we have

$$\int\limits_{R^n} |\psi(x+y)|^{p-\varepsilon} \varkappa\left(\frac{y}{t^e+T^{e'}}\right) dy \le \int\limits_{Z_{t^e+T^{e'}}(x)} |\psi(y)|^{p-\varepsilon} dy$$

$$\le \int\limits_{Z_{(t^{\varkappa})^e+(T^{\varkappa})^{e'}}(x)} |\psi(y)|^{p-\varepsilon} dy \le \|\psi\|^{p-\varepsilon}_{p-\varepsilon, Z_{(t^{\varkappa})^e+(T^{\varkappa})^{e'}}(x)}$$

$$\le \|\psi\|^{p-\varepsilon}_{p), Z_{(t^{\varkappa})^e+(T^{\varkappa})^{e'}}(x)} \varepsilon^{-1} |Z_{(t^{\varkappa})^e+(T^{\varkappa})^{e'}}(x)|$$

$$\le \|\psi\|^{p-\varepsilon}_{p),\varkappa),a,\alpha;Z} \varepsilon^{-1} \prod_{j\in e'} T_j^{\varkappa_j(1+a)-\alpha_j\varepsilon} \prod_{j\in e} t_j^{\varkappa_j(1+a)-\alpha_j\varepsilon}. \tag{1.12}$$

For $y \in V$

$$\int\limits_{U_{\gamma^{\varkappa}}(\overline{x})} |\psi(x+y)|^{p-\varepsilon} dx \le \int\limits_{Z_{\gamma^{\varkappa}}(\overline{x}+y)} |\psi(x)|^{p-\varepsilon} dx \le \|\psi\|^{p-\varepsilon}_{p-\varepsilon, Q_{\gamma^{\varkappa}}(\overline{x}+y)}$$

$$\le \|\psi\|^{p-\varepsilon}_{p), Z_{\gamma^{\varkappa}}(\overline{x}+y)} \varepsilon^{-1} |Z_{\gamma^{\varkappa}}(\overline{x}+y)|$$

$$\le \|\psi\|_{p),\varkappa),a,\alpha;Z} \varepsilon^{-1} \prod_{j\in e_n} \gamma_j^{\varkappa_j(1+a)-\alpha_j\varepsilon}, \tag{1.13}$$

$$\int\limits_{R^n} \left| M_1\left(\frac{y}{t^e+T^{e'}}\right)\right|^s dy = \|M_1\|_s^s \prod_{j\in e'} T_j \prod_{j\in e} t_j. \tag{1.14}$$

It follows from (1.9)-(1.14) for $r = q$ that

$$\|\varphi(\cdot, t^e+T^{e'})\|_{q-\varepsilon, U_{\gamma^{\varkappa}}(\overline{x})} \le C\|M_1\|_s \|\psi\|_{p),\varkappa),a,\alpha;Z} \varepsilon^{-\frac{1}{q-\varepsilon}} \prod_{j\in e_n} \gamma_j^{\frac{\varkappa_j(1+a)-\alpha_j\varepsilon}{q-\varepsilon}}$$

$$\times \prod_{j\in e'} T_j^{1-(1-\varkappa_j-\varkappa_j a+\alpha_j\varepsilon)\left(\frac{1}{p-\varepsilon}-\frac{1}{q-\varepsilon}\right)} \prod_{j\in e} T_j^{1-(1-\varkappa_j-\varkappa_j a+\alpha_j\varepsilon)\left(\frac{1}{p-\varepsilon}-\frac{1}{q-\varepsilon}\right)}. \tag{1.15}$$

Unseating this inequality in (1.8), for $\overline{x} \in U$, we obtain

$$\|A^e_\eta\|_{q-\varepsilon, U_{\gamma^{\varkappa}}(\overline{x})} \le C\|\psi\|_{p),\varkappa),a,\alpha;Z} \varepsilon^{-\frac{1}{q-\varepsilon}} \prod_{j\in e_n} \gamma_j^{\frac{\varkappa_j(1+a)-\alpha_j\varepsilon}{q-\varepsilon}}$$

$$\times \prod_{j\in e'} T_j^{1-(1-\varkappa_j-\varkappa_j a+\alpha_j\varepsilon)\left(\frac{1}{p-\varepsilon}-\frac{1}{q-\varepsilon}\right)} \prod_{j\in e} \eta_j^{\mu_j} \quad (\mu_j > 0).$$

Similarly, we can prove (1.7).

Q.E.D.

2 Main results

We prove two theorems on the properties of functions from the space $S^l_{p),\varkappa),a,\alpha}W(G)$.

Theorem 2.1. Let open bounded set $G \subset R^n$ satisfy the condition of flexible horn [4]; $1 < p < q \leq \infty$, $\varkappa_j \leq \frac{1}{1+a}$, $\nu = (\nu_1, \ldots, \nu_n)$, $\nu_j \geq 0$ are integers; $\mu_j > 0$ $(j \in e_n)$ and let $f \in S^l_{p),\varkappa),a,\alpha}W(G)$.

Then $D : S^l_{p),\varkappa),a,\alpha}W(G) \hookrightarrow L_{q-\varepsilon}(G) \quad (0 < \varepsilon < s_m)$ and it is valid the inequality

$$\|D^\nu f\|_{q-\varepsilon,G} \leq C(\varepsilon) \sum_{e\subseteq e_n} \prod_{j\in e_n} T_j^{s_{e,j}} \|D^{l^e} f\|_{p),\varkappa),a,\alpha;G} \tag{2.1}$$

here

$$s_{e,j} = \begin{cases} \mu_j, \quad j \in e \\ -\nu_j - (1 - \varkappa_j - \varkappa_j a + \alpha_j \varepsilon)\left(\frac{1}{p-\varepsilon} - \frac{1}{q-\varepsilon}\right), \ j \in e' \ . \end{cases}$$

In particular, if

$$\mu_{j,0} = l_j - \nu_j - (1 - \varkappa_j - \varkappa_j a + \alpha_j \varepsilon)\frac{1}{p-\varepsilon} > 0 \ \ (j \in e_n),$$

then $D^\nu f$ is continuous on G and

$$\sup_{x\in G}|D^\nu f(x)| \leq C(\varepsilon) \sum_{e\subseteq e_n} \prod_{j\in e_n} T_j^{s_{e,j,0}} \|D^{l^e} f\|_{p),\varkappa),a,\alpha;G}, \tag{2.2}$$

where $0 < T_j \leq d_j$ $(j \in e_n)$, $C(\varepsilon)$ is a constant independent of f and $T = (T_1, \ldots, T_n)$.

Proof. In conditions of our theorem there exist generalized derivatives $D^\nu f$. Indeed, if $\mu_j > 0$ $(j \in e_n)$, and $l_j - \nu_j > 0$ $(j \in e_n)$. Since $p < q$, $\varkappa_j \leq \frac{1}{1+a}$ $(j \in e_n)$ and

$$S^l_{p),\varkappa),a,\alpha}W(G) \hookrightarrow S^l_{p)}W(G) \hookrightarrow S^l_{p-\varepsilon}W(G) \quad (p - \varepsilon > 1).$$

Then $D^\nu f$ exists on G and belong to $L_{p-\varepsilon}(G)$, and for almost each point $x \in G$ the integral identity received in [4]:

$$D^\nu f(x) = \sum_{e\subseteq e_n} (-1)^{|\nu|} \prod_{j\in e'} T_j^{-1-\nu_j} \int\limits_{O^e}^{T^e} \int\limits_{R^n} \prod_{j\in e} t_j^{l_j-\nu_j-2}$$
$$\times K_e^{(\nu)}\left(\frac{y}{t^e + T^{e'}},\ \frac{\rho(t^e + T^{e'}, x)}{t^e + T^{e'}},\ \rho'(t^e + T^{e'}, x)\right) D^{l^e} f(x+y)dydt^e, \tag{2.3}$$

$0 < T_j \leq d_j \ \ (j \in e_n),\ K_e(\cdot, y, z) \in C_0^\infty(R^n)$ and suppose that $|K_e(x,y,z)| \leq c|K_e^1(x)|$, $e \subseteq e_n$.

Recall that the flexible horn $x + V \subset G$ is the support of the representation (2.3). Based on the Minkowski's inequality, we arrive at

$$\|D^\nu f\|_{q-\varepsilon,G} \leq \sum_{e\subseteq e_n} \prod_{j\in e'} T_j^{-1-\nu_j} \|A_T^e\|_{q-\varepsilon,G}. \tag{2.4}$$

By means of inequality (1.6) for $U = G$, $D^{l^e} f = \psi$, $K_e^{(\nu)} = M$, $\eta_j = T_j$ $(j \in e_n)$ we get inequality (2.1).

Now let conditions $\mu_{j,0} > 0$ $(j \in e_n)$ be satisfied, then based around identity (2.3) from inequality (2.4) we get

$$\|D^\nu f - f_T^{(\nu)}\|_{\infty,G} \leq C(\varepsilon) \sum_{\varnothing \neq e \subseteq e_n} \prod_{j \in e'} T_j^{-1-\nu_j} \|D^{l^e} f\|_{p),\varkappa),a,\alpha;G}.$$

As $T_j \to 0$ $(j \in e_n)$, the left side of this inequality tends to zero, since $f_T^{(\nu)}$ is continuous on G, in our case the convergence in $L_\infty(G)$ coincides with uniform convergence. Then the limit function $D^\nu f$ is continuous on G. The theorem is proved.

Let ξ is n dimension vector. Q.E.D.

Theorem 2.2. Let all the condition of Theorem 2.1. If $\mu_j > 0$ $(j \in e_n)$, then $D^\nu f$ satisfies the Hölder condition in the metric $L_{q-\varepsilon}(G)$ with exponent σ_j, more exactly

$$\|\Delta(\xi, G) D^\nu f\|_{q-\varepsilon,G} \leq C(\varepsilon) \|f\|_{S^l_{p),\varkappa),a,\alpha} W(G)} \prod_{j \in e_n} |\xi_j|^{\sigma_j}, \tag{2.5}$$

where σ_j is an arbitrary number satisfying the inequalities:

$$\begin{aligned} &0 \leq \sigma_j \leq 1, \quad \text{if} \quad \mu_j > 1, \;\; j \in e, \\ &0 \leq \sigma_j < 1, \;\; \text{if} \;\; \mu_j = 1, \;\; j \in e, \;\;\; 0 \leq \sigma_j \leq 1, \;\; j \in e', \\ &0 \leq \sigma_j \leq \mu_j, \;\; \text{if} \;\; \mu_j, 1, \;\; j \in e. \end{aligned} \tag{2.6}$$

If $\mu_{j,0} > 0$ $(j \in e_n)$, then

$$\sup_{x \in G} |\Delta(\xi, G) D^\nu f(x)| \leq C(\varepsilon) \|f\|_{S^l_{p),\varkappa),a,\alpha} W(G)} \prod_{j \in e_n} |\xi_j|^{\sigma_{j,0}}$$

where $\sigma_{j,0}$ satisfy the same conditions as σ_j, with $\mu_{j,0}$ instead of μ_j.

Proof. By Lemma 8.6 of [2], there *ia* domain $G_u \subset G$ $(u = (u_1, ..., u_n)$, $u_j = s_j r(x)$, $s_j > 0$, $j \in e_n, r(x) = \text{dist}(x, \partial G), x \in G)$. Suppose that $|\xi_j| < u_j$, $(j \in e_n)$. Then, for every $x \in G_u$, then segment joining the points x and $x + \xi$ is constained in G. Consequently, for all points of this segment, identifies (2.3) with the same kernels are valid. After transformations, from (2.3) we obtain

$$|\Delta(\xi,G)D^{\nu}f(x)| \le C_1 \sum_{e\subseteq e_n}(-1)^{|\nu|}\prod_{j\in e'}T_j^{-1-\nu_j}\int\limits_{O^e}^{|\xi^e|}\prod_{j\in e}t_j^{l_j-\nu_j-2}dt^e$$
$$\times\int\limits_{R^n}\left|K_e^{(\nu)}\left(\frac{y}{t^e+T^{e'}},\ \frac{\rho(t^e+T^{e'},x)}{t^e+T^{e'}},\ \rho'(t^e+T^{e'},x)\right)\right|$$
$$\times|\Delta(\xi,G)D^{l^e}f(x+y)|dy + C_2\sum_{e\subseteq e_n}(-1)^{|\nu|}\prod_{j\in e'}T_j^{-2-v_j}\prod_{j\in e_n}|\xi_j|\int\limits_{|\xi^e|}^{T^e}\prod_{j\in e}t_j^{l_j-\nu_j-3}dt^e$$
$$\times\int\limits_{R^n}\left|K_e^{(\nu+1)}\left(\frac{y}{t^e+T^{e'}},\frac{\rho(t^e+T^{e'},x)}{t^e+T^{e'}},\ \rho'(t^e+T^{e'},x)\right)\right|$$
$$\times\int\limits_0^1\ldots\int\limits_0^1|D^{l^e}f(x+y+\xi_1\omega_1+\ldots+\xi_n\omega_n)|d\omega dy$$
$$= C_1\sum_{e\subseteq e_n}E_1^e(x,\xi)+C_2\sum_{e\subseteq e_n}E_2^e(x,\xi), \qquad (2.7)$$

where $0<T_j\le\min\{1,d_j\}$ $(j\in e_n)$; $|\xi^e|=(|\xi_1^e|,\ldots,|\xi_n^e|)$, $|\xi_j^e|=|\xi_j|$ for $j\in e$; $|\xi_j^e|=0$ for $j\in e'$. Assume that $|\xi_j|<T_j$ $(j\in e_n)$, and consequently $|\xi_j|\le\min\{u_j,T_j\}$ $(j\in e_n)$. If $x\in G\setminus G_u$, then by definition $\Delta(\xi,G)D^v f(x)=0$.

By (2.7)

$$\|\Delta(\xi,G)D^{\nu}f\|_{q-\varepsilon,G}=\|\Delta(\xi,G)D^{\nu}f\|_{q-\varepsilon,G_u}$$
$$\le C_1\sum_{e\subseteq e_n}|E_1^e(\cdot,\xi)\|_{q-\varepsilon,G_u}+C_2\sum_{e\subseteq e_n}\|E_2^e(\cdot,\xi)\|_{q-\varepsilon,G_u}. \qquad (2.8)$$

By means of inequality (1.6) for $D^{l^e}f=\psi$, $\eta_j=|\xi_j|$ $(j\in e)$ we obtain

$$\|E_2^e(\cdot,\xi)\|_{q-\varepsilon,G_u}\le C_1(\varepsilon)\|D^{l^e}f\|_{p),\varkappa),a,\alpha;G}\prod_{j\in e}\|\xi_j\|^{\mu_j}, \qquad (2.9)$$

and by means of inequality (1.7) for $D^{l^e}f=\psi$, $\eta_j=|\xi_j|$ $(j\in e)$ we obtain

$$\|E_2^e(\cdot,\xi)\|_{q-\varepsilon,G_u}\le C_2(\varepsilon)\|D^{l^e}f\|_{p),\varkappa),a,\alpha;G}\prod_{j\in e}\|\xi_j\|^{\mu_j}. \qquad (2.10)$$

From inequalities (2.8)-(2.10) we obtain the required inequality.

Now suppose that $\{\xi_j\}\ge\min\{u_j,T_j\}$. Then

$$\|\Delta(\xi,G)D^{\nu}f\|_{q-\varepsilon,G}\le 2\|D^{\nu}f\|_{q-\varepsilon,G}\le C(u,T)\|D^{\nu}f\|_{q-\varepsilon,G}\prod_{j\in e_n}\|\xi_j\|^{\sigma_j}.$$

Estimating $\|D^{\nu}f\|_{q-\varepsilon,G}$ by means of inequality (2.1), we obtain the sought inequality in this case as well. The Theorem is proved. Q.E.D.

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